MATH 301: MATHEMATICAL METHODS II
2024/2025 SESSIONS.

(1a) Fourier Series: A function $f(x)$ is said to be periodic function if there is some positive number T such that $f(x+T) = f(x)$ for all x, where T is called a period of $f(x)$ i.e $f(x)$ is repeated after an interval of independent variable x.

Given $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx, \quad P = \pi \Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} dx\right] = \frac{1}{\pi}\int_{0}^{\pi} dx = \frac{1}{\pi}[x]_0^{\pi}$$

$$= \frac{1}{\pi}[\pi - 0] = \frac{\pi}{\pi} = 1$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\cos\frac{n\pi x}{P}dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \frac{1}{\pi}\frac{\sin nx}{n}\Big|_0^{\pi}$$

$$= \frac{1}{n\pi}[\sin\pi n - \sin 0] = \frac{1}{n\pi}\sin\pi n = 0, \text{ since } \sin(\pi n) = 0$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x)\sin\frac{n\pi}{P}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right] = \frac{1}{\pi}\left[\int_{0}^{\pi}\sin nx\,dx\right] = \frac{-1}{\pi n}\cos nx\Big|_0^{\pi}$$

$$= \frac{-1}{\pi n}[\cos n\pi - \cos 0] = \frac{-1}{n\pi}[\cos n\pi - 1] = \frac{1}{n\pi}[1 - \cos n\pi]$$

$$= \frac{1}{n\pi}[1 - (-1)^n] \text{ where } \cos n\pi = (-1)^n$$

Hence, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{P} + b_n\sin\frac{n\pi x}{P}\right)$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left(0\cdot\cos\frac{n\pi x}{\pi} + \frac{1}{n\pi}[1 - \cos n\pi]\sin\frac{n\pi x}{\pi}\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{n\pi}(1 - (-1)^n)\sin nx\right)$$ as the required fourier series expansion of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

(1b) Given $f(x) = e^x$ on $(0,\pi)$.

Fourier Sine Series on $(0,\pi)$ $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}$ where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx = \frac{2}{\pi}\int_0^\pi e^x \sin\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^\pi e^x \sin nx\,dx$$

$$= \frac{2n}{\pi(1+n^2)}\left[e^\pi(-1)^{n+1}+1\right]$$

$$f(x) = e^x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2n}{\pi(1+n^2)}\left[e^\pi(-1)^{n+1}+1\right]\sin nx.$$

Fourier Cosine Series on $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} \quad \text{where}$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx$$

Now, $a_0 = \frac{2}{\pi}\int_0^\pi e^x\,dx = \frac{2}{\pi}e^x\Big|_0^\pi = \frac{2}{\pi}(e^\pi - 1)$

$$\Rightarrow \frac{a_0}{2} = \frac{e^\pi - 1}{\pi} \quad \text{—— (1)}$$

$$a_n = \frac{2}{\pi}\int_0^\pi e^x \cos\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^\pi e^x \cos nx\,dx$$

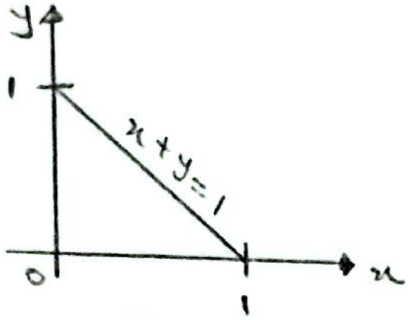
$$= \frac{2}{\pi}\left[\frac{e^\pi(-1)^n}{1+n^2} - \frac{1}{1+n^2}\right] = \frac{2}{\pi}\left[\frac{e^\pi(-1)^n - 1}{1+n^2}\right]$$

$$f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p}$$

$$= \frac{e^\pi - 1}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{e^\pi(-1)^n - 1}{1+n^2}\right]\cos nx$$

(2a) i: $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant $x+y \le 1$.

The region of integration R is the area bounded by the straight line $x+y=1$ and the coordinate axes i.e $x=0$, $x=1-y$, $y=0$ and $y=1$



$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{y=1}\int_0^{1-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_0^{1-y} dy = \int_0^1 \left[\frac{(1-y)^3}{3} + y^2(1-y)\right] dy$$

$$= -\frac{(1-y)^4}{12} + \frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1$$

$$= \left[-\frac{(1-1)^4}{12} + \frac{1}{3} - \frac{1}{4}\right] - \left[-\frac{(1-0)^4}{12}\right] = \left[\frac{1}{3} - \frac{1}{4}\right] - \left[-\frac{1}{12}\right] = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = 1/6$$

ii) $$\int_0^1\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^1\int_0^x\int_0^{x+y} e^{x+y}\cdot e^z\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x e^{x+y}\left[e^{x+y} - e^0\right] dy\,dx = \int_0^1\int_0^x e^{x+y}\left[e^{x+y} - 1\right] dy\,dx = \int_0^1\int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy\,dx$$

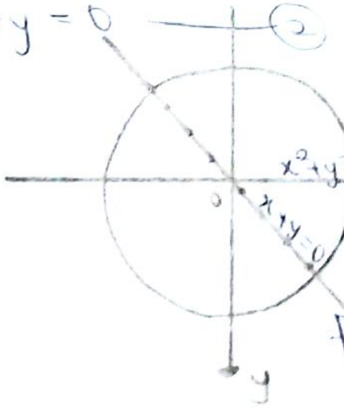
$$= \int_0^1\int_0^x \left[e^{2x}\cdot e^{2y} - e^x\cdot e^y\right] dy\,dx = \int_0^1 \left[e^{2x}\frac{e^{2y}}{2}\Big|_0^x - e^x e^y\Big|_0^x\right] dx$$

$$= \int_0^1 \left[\frac{e^{2x}}{2}\left(e^{2x} - e^0\right) - e^x\left(e^x - e^0\right)\right] dx = \int_0^1 \left[\frac{e^{2x}}{2}\left(e^{2x} - 1\right) - e^x\left(e^x - 1\right)\right] dx$$

$$= \int_0^1 \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\right] dx = \int_0^1 \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x\right) dx = \frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\Big|_0^1$$

$$= \left(\frac{e^4}{8} - \frac{3e^2}{4} + e^1\right) - \left(\frac{1}{8} - \frac{3}{4} + 1\right) = \frac{e^4}{8} - \frac{3e^2}{4} + e - \frac{3}{8}$$

2b) The equation of the circle is $x^2+y^2=25$ — (1) and the line is $x+y=0$ — (2) from eqn (1), the circle $x^2+y^2=25$ is centered at the origin with radius 5 and the line $x+y=0$ passes through the origin which is equivalent to $y=-x$, a diagonal line that split the circle into two equal halves.



From eqn (1) and (2), On eliminating y, we have

$y=-x$ and $x^2+(-x)^2=25 \Rightarrow x^2+x^2=25$

$\Rightarrow 2x^2 = 25 \Rightarrow 2x^2-25=0$

$\Rightarrow 2\left(x^2-\frac{25}{2}\right)=0 \Rightarrow 2\left[\left(x-\frac{5}{\sqrt{2}}\right)\left(x+\frac{5}{\sqrt{2}}\right)\right]=0$

$\Rightarrow x = 5/\sqrt{2}$ or $x = -5/\sqrt{2}$ Hence, $y = -5/\sqrt{2}$ or $y = 5/\sqrt{2}$ respectively. (3)

Thus, the point of intersection of the diagonal are $A(5/\sqrt{2}, -5/\sqrt{2})$ and $B(-5/\sqrt{2}, 5/\sqrt{2})$.
So, the required area is the area lying between the diagonal $y = -x$ and $y^2 = 25 - x^2$ $(y = \sqrt{25-x^2})$, $x = -5/\sqrt{2}$ and $x = 5/\sqrt{2}$.

Hence, Area $= \int_{-5/\sqrt{2}}^{5/\sqrt{2}} \int_{-x}^{\sqrt{25-x^2}} dy\,dx = \int_{-5/\sqrt{2}}^{5/\sqrt{2}} y \Big|_{-x}^{\sqrt{25-x^2}} dx = \int_{-5/\sqrt{2}}^{5/\sqrt{2}} [\sqrt{25-x^2} + x]\,dx$

$= \frac{1}{2}\left(x\sqrt{5^2-x^2} + 5^2 \sin^{-1}(x/5)\right)] + \frac{x^2}{2}\Big|_{-5/\sqrt{2}}^{5/\sqrt{2}} = \left[\frac{1}{2}\left(\frac{25}{2} + \frac{25\pi}{4}\right) + \frac{5}{4}\right] -$
$\left[\frac{1}{2}\left(-\frac{25}{2} - \frac{25\pi}{4}\right) + \frac{5}{4}\right] = \frac{1}{2}\left(\frac{25}{2} + \frac{25\pi}{4}\right) - \frac{1}{2}\left(-\frac{25}{2} - \frac{25\pi}{4}\right)$
$= \frac{1}{2}\left[\frac{25}{2} + \frac{25}{2} + \frac{25\pi}{4} + \frac{25\pi}{4}\right] = \frac{1}{2}\left(25 + \frac{25}{2}\pi\right) = \frac{25}{4}(2 + \pi)$
$= \frac{25}{4}(\pi + 2)$

3a) Since $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ is defined along a curve $C: x = f(t), y = g(t), z = h(t), a \le t \le b$, and $\vec{r}(t) = f(\vec{t})i + g(\vec{t})j + h(\vec{t})k$
then, the derivative of $\vec{r}(t) = \frac{d\vec{r}(t)}{dt} = f'(\vec{t})i + g'(\vec{t})j + h'(\vec{t})k$ so that
$d\vec{r}(t) = df(\vec{t})i + dg(\vec{t})j + dh(\vec{t})k$.
We can write $\vec{F}(x,y,z)d\vec{r}(t) = (P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k)(df(t)i + dg(t)j + dh(t)k)$ —— (1)

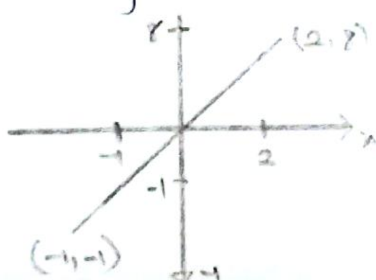
And therefore,
$\int \vec{F}(x,y,z)d\vec{r} = \int_C P(x,y,z)df(t) + Q(x,y,z)dg(t) + R(x,y,z)dh(t)$
$= \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ —— (2)

Now, $\frac{d\vec{r}}{dt} = \frac{dr}{ds} \cdot \frac{ds}{dt}$ if $\frac{ds}{dt} = T$ a unit vector along the tangent to C, we have $d\vec{r} = T\,ds$

Hence, eqn (2) becomes
$\int \vec{F}(x,y,z)d\vec{r} = \int_C \vec{F} d\vec{r} = \int_C \vec{F} \cdot T\,ds$

Thus, $\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_C F \cdot dr$ □.

(b) $\int_C xy\,dx + x^2 dy$, where $C: y = x^3, -1 \le x \le 2$



from $y = x^3$, we have $dy = 3x^2 dx$
$\therefore \int_C xy\,dx + x^2 dy = \int_{-1}^{2} x(x^3)dx + x^2(3x^2)dx$
$= \int_{-1}^{2} x^4 dx + 3x^4 dx = \int_{-1}^{2} 4x^4 dx$

$$= \frac{4x^5}{5}\Big|_{-1}^{2} = \frac{4}{5}[2^5-(-1)^5] = \frac{4}{5}(33) = \frac{132}{5}$$

4ai) The divergence of a function is a scalar quantity $\nabla\cdot\vec{v}$ of the vector field $\vec{v}$ denoted by div $\vec{v}$ and defined by div $\vec{v} = \nabla\cdot\vec{v}$ $= (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\vec{v}$. If $\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ then;

$$\text{div } \vec{v} = \nabla\cdot\vec{v} = \frac{\partial\vec{v}_1}{\partial x} + \frac{\partial\vec{v}_2}{\partial y} + \frac{\partial\vec{v}_3}{\partial z}$$

ii) If $\vec{v}(x,y,z)$ is a continuous differentiable vector point function, then the curl is a cross product denoted by Curl $\vec{v}$ and is defined by Curl $\vec{v} = \nabla\times\vec{v}$. If $\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ then;

$$\text{Curl } \vec{v} = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\times(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{vmatrix} = i\left(\frac{\partial\vec{v}_3}{\partial y} - \frac{\partial\vec{v}_2}{\partial z}\right) - j\left(\frac{\partial\vec{v}_3}{\partial x} - \frac{\partial\vec{v}_1}{\partial z}\right) + k\left(\frac{\partial\vec{v}_2}{\partial x} - \frac{\partial\vec{v}_1}{\partial y}\right)$$

4bi) $v(x,y,z) = (xy\sin z)i + (y^2\sin x)j + (z^2\sin xy)k$

$$\text{div } \vec{v} = \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}(y^2\sin x) + \frac{\partial}{\partial z}(z^2\sin xy)$$

$$= y\sin z + 2y\sin x + 2z\sin xy \quad \text{at } (0,\pi,\pi)$$

$$\text{div } \vec{v} = \pi\cancel{\sin\pi}^{0} + 2\pi\sin 0 + 2\pi\sin 0 = \underline{\underline{0}}$$

ii) $$\text{Curl } \vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy\sin z & y^2\sin x & z^2\sin xy \end{vmatrix} = i\left(\frac{\partial}{\partial y}(z^2\sin xy) - \frac{\partial}{\partial z}(y^2\sin x)\right) - j\left(\frac{\partial}{\partial x}(z^2\sin xy) - \frac{\partial}{\partial z}(xy\sin z)\right] + k\left(\frac{\partial}{\partial x}(y^2\sin x) - \frac{\partial}{\partial y}(xy\sin z)\right]$$

$$= i(2z\sin xy) - j(yz^2\cos xy - xy\cos z) + k(y^2\cos x - x\sin z) \quad \text{at } (0,\pi,\pi)$$

$$\text{Curl } \vec{v} = i(2\pi\sin 0) - j(\pi^3\cos 0 - 0) + k(\pi^2\cos 0 - 0)$$

$$= \pi^3 j + \pi^2 k = \pi^2(\pi j + k)$$

5ai) If $f(x,y)$ or $f(x,y,z)$ is a scalar function defined and differential at each point (x,y) or (x,y,z) in a certain region then the gradient of f denoted by "∇f or grad f" is defined as;

$$\nabla f = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})f.$$

ii) The directional derivative of a vector function $f(x,y)$ or $f(x,y,z)$ of a point p in the direction of a vector gives the rate of change of f in an arbitrary direction in space denoted by $D\hat{u}f$ and defined by;

$D_{\hat{u}}f = \text{grad} f \cdot \hat{u} = \nabla f \cdot \hat{u} = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|}$.

(b) $F(x,y,z) = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of the vector $i + 2j + 2k$.

$\vec{u} = i + 2j + 2k \Rightarrow |\vec{u}| = \sqrt{1^2 + 2^2 + 2^2} = 3$

Thus, $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{i + 2j + 2k}{3} = \frac{1}{3}(i + 2j + 2k)$.

Also,

$$\nabla f = i\frac{\partial}{\partial x}(xy^2 + yz^3) + j\frac{\partial}{\partial y}(xy^2 + yz^3) + k\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= i(y^2) + j(2xy + z^3) + k(3yz^2) \text{ at } (2,-1,1)$$

$$= i(-1)^2 + j(2(2)(-1) + (1)^3] + k(3(-1)(1)^2] = i - 3j - 3k$$

Hence;

$$D_{\hat{u}}f = \nabla f \cdot \hat{u} = (i - 3j - 3k) \cdot \frac{1}{3}(i + 2j + 2k) = \frac{1}{3}(1 - 6 - 6) = \frac{-11}{3}$$

(6a.) Green's theorem states that if $P, Q, \frac{\partial P}{\partial y}$, and $\frac{\partial Q}{\partial x}$ are continuous on R, then;

$$\oint P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

To evaluate $\oint_C (x - 3y)dx + (4x + y)dy$ where C is the rectangle with vertices at $(-2,0), (3,0), (3,2)$ and $(-2,2)$

$P(x,y) = x - 3y$ and $Q(x,y) = 4x + y \Rightarrow \frac{\partial P}{\partial y} = -3$ and $\frac{\partial Q}{\partial x} = 4$

Thus, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4 - (-3) = 4 + 3 = 7$

Then, the rectangle is $x: -2$ to 3, $y: 0$ to 2

$$\text{Thus, } \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R 7\,dxdy = \int_{-2}^{3}\int_0^2 7\,dy\,dx = \int_{-2}^{3} 7y\Big|_0^2 dx$$

$$= 7\int_{-2}^{3}(2 - 0)dx = 7\int_{-2}^{3} 2\,dx = \int_{-2}^{3} 14\,dx = 14x\Big|_{-2}^{3} = 14[3 - (-2)] = 14(5)$$

$$= 70$$

(b.) Stoke's theorem states that the surface integral of the component of curl $\vec{f}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{f}$ taken along the closed curved C.

Mathematically,

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds$$ where

$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds.